

Stress Testing CDOs

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Abstract

Analyses regarding the responsibility of risk management for the current credit crises have found a lack of stress tests as one important issue. In this article, we argue that stress tests are even more important a risk management tool with structured finance products like CDOs. We explain why the specific risk profile of such assets requires a dynamic modeling. In an extensive case study a stress test comparison is made between portfolios including conventional bonds and structured products. The results clearly show the increased risk contribution of structured products which reveals explicitly only in a dynamic view.

KEYWORDS: *Stress Testing, CDO, Basel II, 1-Factor Model, Dynamics*

1 Introduction

The sustained financial crises has triggered investigations on potential weaknesses of risk management practices having contributed to what happened (SSG 2008). In this field, a lack of stress testing is a frequently identified issue and so it is a hot topic on the Basel II agenda now.

This lack does not only apply to practice but also to research. While the general literature on stress testing is quite extensive (e.g. Berkowitz 2000, Kupiec 1998, Roesch & Scheule 2007, Simons & Rolwes 2008) there are hardly any contributions in terms of credit derivatives (Dewyspelaere et al. 2004).

Why should a lack of stress tests have such tremendous impact just now? Going back in history we find a series of “financial crises” and bursting bubbles but no one was as severe and threatening as the current one. Were risk methods deteriorating? We think “partly”. In our opinion one major factor at the root of the crisis is the explosive growth of stress-sensitive financial products which did not come along with enhanced stress testing procedures. It is well known that structured products are more sensitive to changes in systematic risk. As a result, they are more sensitive to stress tests and so stressing systematic risk factors has considerable impact with them. Hence, our major objective in this article is to show that stress tests are more relevant with structured products than with bonds and to identify the major drivers of risk. Furthermore, we suggest a dynamic modeling approach for two reasons. First, systematic factors are known to have dynamic persistence, i.e., there are phases of better and phases of worse economic conditions. Thus, increased systematic dependence requires accurate modeling of the systematic factor. Second, the common approach of single-period “through-the-cycle” modeling averages out important extreme scenarios. As transition from low to high risk is rather abrupt than gradual with structured products multiple-periods “point-in-time” modeling of the systematic factor is important in order to discover adverse scenarios.

While the literature on stress testing is comparatively large, hardly any of them are on CDOs or on portfolios including CDOs. Fender et al. (2008) present a very limited stress analysis of the risk contributions of CDOs. Similarly, the article of Dewyspelaere et al. (2004) is limited in the set of considered risk parameters, includes no dynamic view and finally provides no model details. In the present article, we go a step further and present a fully dynamic risk analysis of CDOs and their risk contributions to a portfolio under stress. Most importantly, increased dynamic risk and stress-sensitivity are found with CDOs. We identify the most stress-sensitive drivers of CDO risk and analyze their behavior in time. Our analyses substantiate clearly the increased stress-sensitivity of CDOs and the necessity of dynamic modeling in this context.

In the next section we outline the model setup and describe relevant risk

measures. After that, we introduce CDOs and specify a sample asset pool. The following section includes the results of the stress test study.

2 Model Setup

In this section we describe the model setup upon which our analyses are based. We consider a Merton-style Gaussian one-factor model as suggested in the Basel II specification. F represents a common systematic factor which affects all obligors' default probabilities.

2.1 Single Period Gaussian 1-Factor Model

In a portfolio of $i = 1, \dots, n$ obligors default of i is modeled as a threshold event:

$$D_i = \mathbf{1}_{\{R_i < c_i\}} \quad (1)$$

where R_i is a random variable comprising two terms

$$R_i = \sqrt{\rho}F + \sqrt{1 - \rho}U_i \quad (2)$$

a common (systematic) factor F and an idiosyncratic factor U_i . Both are iid standard normal and so is R_i . D_i is a default indicator which jumps to unity if R_i falls below c_i . As a result, the probability of default (PD) of obligor i is $\mathbb{P}[D_i = 1] = \lambda_i = \Phi(c_i)$, where Φ denotes the standard Gaussian cumulative distribution function.

Conditioning on F , the specific structure of R_i with a common and an idiosyncratic factor implies independence of any two $R_i, R_{i'}, i, i' \in \{1, \dots, n\}, i \neq i'$. The conditional probability of default of obligor i is

$$\lambda_i(F) = \Phi\left(\frac{\Phi^{-1}(\lambda_i) - \sqrt{\rho}F}{\sqrt{1 - \rho}}\right) \quad (3)$$

Now let N_i denote the notional of obligor i . Then, portfolio loss is given by

$$L = \frac{1}{N} \sum_i \mathbf{1}_{\{D_i=1\}} \cdot N_i \cdot \text{LGD}_i \quad (4)$$

where $N = \sum_i N_i$ denotes total exposure.

2.2 Multiple Periods Gaussian 1-Factor Model

So far the described approach is standard and frequently applied in practice. Our next step is to extend the model to multiple periods. This is most easily done assuming stochastic processes for F and U_i , i.e., F_t and U_{it} . In the absence of any other hypothesis we simply assume that U_{it} is iid standard normal.

However, the systematic term admits more structure. Empirical research shows that default rates have a cyclical behavior and persistence phases. Hence, we specify F_t in one of the most simple ways as first-order autoregressive process AR(1)¹:

$$F_t = \alpha F_{t-1} + \sigma W_t \quad (5)$$

where α and σ are parameters, F_0 is an initial value of the process and W_t is iid standard normal. Furthermore, we set $\sigma = \sqrt{1 - \alpha^2}$ so that $F_t \rightarrow \mathcal{N}(0, 1)$ as t grows. The first two moments of the unconditional process F_t are $\mathbb{E}[F_t] = 0$ and $\mathbb{V}[F_t] = 1$. Given F_0 we have $\mathbb{E}[F_t] = \alpha^t F_0$ and $\mathbb{V}[F_t] = \sigma^2 \sum_{j=0}^{t-1} \alpha^{2j} = 1 - \alpha^{2t}$.

A multi-period setting requires evaluation of the threshold model once per period among the survivors. The default indicators are defined accordingly as follows

$$D_{it} = \begin{cases} 1 & \text{if } R_{it} < c_{it} \\ 0 & \text{else} \end{cases} \quad (6)$$

In each period, $i \in N_t$, i.e., only names that have not defaulted yet are kept in the asset pool. The set of survivors of period $t - 1$ is defined as $N_t = \{i : D_{it'} = 0, t' < t\}$.

Now, substituting F_t in R_{it} we obtain

$$R_{it} = \sqrt{\rho} \alpha^t F_0 + \sqrt{\rho} \sigma \sum_{j=0}^{t-1} \alpha^j W_{t-j} + \sqrt{1 - \rho} U_{it} \quad (7)$$

which clearly shows the exponentially decreasing weight of systematic risk disturbances of earlier periods.

The default thresholds c_{it} have to be adapted to the hazard rate term structures (λ_{it}). Thus, for any $t > 1$ we set

$$\mathbb{P}[R_{it} < c_{it} \mid R_{it'} > c_{it'}, t' < t] = \lambda_{it} \quad (8)$$

¹This type of process has also been used by Lamb et al. (2008) and McNeil & Wendin (2006).

and solve for c_{it} . For $t = 1$ the condition drops.

The marginal and joint distributions of R_{it} and $R_{it'}, t' < t$, are Gaussian with expectation

$$(\mathbb{E}[R_{i1}], \dots, \mathbb{E}[R_{it}]) = (0, \dots, 0) \quad (9)$$

and covariances

$$\begin{aligned} Cov(R_{it}, R_{it'}) &= \mathbb{E}[R_{it} \cdot R_{it'}] - \mathbb{E}[R_{it}]\mathbb{E}[R_{it'}] \\ &= \rho\alpha^{|t-t'|} + (1 - \rho) \mathbf{1}_{\{t=t'\}} \end{aligned} \quad (10)$$

The derivation of the covariance matrix can be found in the appendix. The above hazard rates based on a multidimensional Gaussian are easily calculated either via simulation of the model or numerical integration of (8). As a result, we may derive c_{it} consecutively (given $c_{it'}, t' < t$) in a bootstrap fashion².

Furthermore, we need loss given default, LGD_{it} , the percentage loss in default period t . We use a similar specification as Duellmann & Trapp (2004) who assume the following process for $Y_{it} = \ln \left[\frac{1-LGD_{it}}{LGD_{it}} \right]$

$$Y_{it} = \mu + \tilde{\sigma}\sqrt{\omega_1}F_t + \tilde{\sigma}\sqrt{1-\omega_1}E_{it} \quad (11)$$

where $i \in D_t$, the index set of defaulted names by the end of t , $D_t = \{i \in N_t : D_{it} = 1\}$ and E_{it} is a name-specific standard normally distributed innovation.

μ and $\tilde{\sigma}$ are linear transformation coefficients and ω_1 controls the influence of F_t . The first two moments are $\mathbb{E}[Y_{it}] = \mu$ and $\mathbb{V}[Y_{it}] = \tilde{\sigma}^2$ without knowledge of F_0 and $\mathbb{E}[Y_{it} | F_0] = \mu + \tilde{\sigma}\sqrt{\omega_1}\alpha^t F_0$ and $\mathbb{V}[Y_{it} | F_0] = \tilde{\sigma}^2 (1 - \omega_1\alpha^{2t})$ given F_0 . The latter conditional moments tend to the unconditional ones as t rises.

Thus, LGD and PD are based on the same process F_t . Ultimately, portfolio loss in period t is given by

$$L_t = \frac{1}{N} \sum_{t' \leq t} \sum_{i \in D_{t'}} N_i \cdot LGD_{it'} \quad (12)$$

where we assume that notionals N_i are time-invariant.

3 CDO Structure and Risk Measures

In this paper we analyze the risk behavior of CDOs under stress. To that end, we have to define a CDO as well as measures to exhibit and quantify

²Inversion is done easily via a one-dimensional root search algorithm.

its risk properties.

3.1 CDOs

A CDO is a securitized pool of defaultable assets where the issued notes relate to different loss layers (“tranches”) of the total notional. A specific tranche incurs losses only in excess of a minimum loss level $A^{(\text{tr})}$ and also only up to a maximum loss level $B^{(\text{tr})}$. As a result, tranches differ in terms of seniority. An example of a possible tranche structure is given in Table 1.

Tranche	$A^{(\text{tr})}$	$B^{(\text{tr})}$
Super Senior	0.17	1.00
Senior	0.11	0.17
Mezzanine	0.08	0.11
Junior	0.05	0.08
Equity	0.00	0.05

Table 1: CDO tranches.

To formalize a tranche, let $0 \leq A^{(\text{tr})} < B^{(\text{tr})} \leq 1$ denote a percentage interval of the asset pool notional $N = \sum_i N_i$. A specific note of this tranche incurs losses if total cumulative asset pool loss exceeds the lower attachment point $A^{(\text{tr})}$ of the tranche, i.e., $L_t > A^{(\text{tr})}$ and a complete default of the CDO occurs if $L_t \geq B^{(\text{tr})}$. Between these extremes CDO tranche loss is given by

$$L_t^{(\text{tr})} = \left[\min(L_t, B^{(\text{tr})}) - \min(L_t, A^{(\text{tr})}) \right] / \left[B^{(\text{tr})} - A^{(\text{tr})} \right] \quad (13)$$

3.2 Sample Asset Pool and CDO

A sample asset pool which will be the foundation of our later analyses is given in Table 2. It is adopted from Bluhm & Overbeck (2007, Appendix 6.9).

Class	#	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
AAA	5	0.0000	0.0000	0.0004	0.0003	0.0005
AA	12	0.0001	0.0002	0.0005	0.0008	0.0010
A	22	0.0004	0.0009	0.0013	0.0017	0.0023
BBB	32	0.0029	0.0057	0.0063	0.0090	0.0090
BB	17	0.0128	0.0271	0.0350	0.0344	0.0318
B	8	0.0624	0.0863	0.0845	0.0752	0.0607
CCC	4	0.3235	0.1478	0.1095	0.0972	0.1260

Table 2: Asset pool described in Bluhm & Overbeck (2007): rating class, number of names, hazard rates for $t = 1, \dots, 5$.

The pool comprises 100 credit risky names from 7 rating classes with different hazard rate term structures. Maturity is $T = 5$ years and portfolio

exposures are homogeneous. A CDO with strict loss prioritization (SLP) rule references the pool³.

Below, our focus is on a mezzanine tranche with attachment point $A^{(\text{tr})} = 0.08$. To study the relevance of tranche width, we look at two alternative detachment points $B^{(\text{tr})} = 0.11$ and $B^{(\text{tr})} = 0.09$. Thus, the first tranche variant has width 3% and the second has 1%.

3.3 Asset pool model

To model the asset pool we employ the dynamic single risk-factor model specified in the previous chapter. Hence, the asset values which are compared with threshold c_{it} are given by

$$R_{it} = \sqrt{\rho^{(\text{AP})}} F_t^{(\text{AP})} + \sqrt{1 - \rho^{(\text{AP})}} U_{it} \quad (14)$$

and the systematic factor driving the asset pool (AP) has first-order autoregressive dynamics

$$F_t^{(\text{AP})} = \alpha F_{t-1}^{(\text{AP})} + \sigma W_t \quad (15)$$

Finally, LGD_{it} is given by

$$\text{LGD}_{it} = \frac{1}{1 + \exp\left(\mu^{(\text{AP})} + \tilde{\sigma} \sqrt{\omega_1} F_t^{(\text{AP})} + \tilde{\sigma} \sqrt{1 - \omega_1} E_{it}\right)} \quad (16)$$

We add a superscript (AP) to indicate that these parameters belong to the asset pool model. In the next section we introduce a second portfolio so that differentiation will be helpful at this place.

3.4 CDO Risk Measures

In order to qualify and quantify the risk behavior of CDO tranches we employ several measures.

Hitting Probability Hitting probability is simply the probability that total asset pool loss exceeds the attachment point, i.e., $\mathbb{P}[L_t > A^{(\text{tr})}]$. This measure indicates the cumulative risk of tranche losses over t periods.

Tranche Hazard Rate The tranche hazard rate is defined as the probability of loss in t given no loss until $t - 1$:

$$\lambda_t^{(\text{tr})} = \frac{\mathbb{P}[L_t > A^{(\text{tr})}, L_{t-1} \leq A^{(\text{tr})}]}{\mathbb{P}[L_{t-1} \leq A^{(\text{tr})}]} \quad (17)$$

³SLP means that tranches are paid in strict order of their seniority.

This measure reveals hitting risks in period t .

Expected LGD Expected loss given default of a tranche, $\mathbb{E}\left[L_t^{(\text{tr})} \mid L_t^{(\text{tr})} > 0\right]$, is a measure of the level of cumulated losses incurred by period t .

Incremental Value at Risk The final risk measure is incremental value at risk of a tranche in period t . It is defined as the difference between the VaR of a “superportfolio” including the tranche and the VaR of the same “superportfolio” without the tranche (e.g. Felsenheimer et al. 2006). Formally,

$$\Delta\text{VaR}_t(q) = \mathbf{F}_{L_t^{(\text{SP})^+}}^{-1}(q) - \mathbf{F}_{L_t^{(\text{SP})}}^{-1}(q) \quad (18)$$

where $\mathbf{F}_{L_t}^{-1}$ denotes the (generalized) inverse of L_t at fractional rank q and $L_t^{(\text{SP})^+}$ and $L_t^{(\text{SP})}$ denote loss of the superportfolio with and without the CDO tranche, respectively. Note that $\Delta\text{VaR}_t(q)$ is generally not portfolio-invariant. As shown by Gordy (2003), in the case of a perfectly diversified portfolio and a single systematic risk factor, $\text{VaR}_t(q)$ and thus $\Delta\text{VaR}_t(q)$ are portfolio-invariant, but we do not rely on this result here.

For the superportfolio we assume $N = 500$ names with identical PDs. The PD term structure is a weighted average of the asset pool PD term structure. The asset values are again modeled as

$$R_{it} = \sqrt{\rho^{(\text{SP})}} F_t^{(\text{SP})} + \sqrt{1 - \rho^{(\text{SP})}} U_{it} \quad (19)$$

but now carry superscript (SP) to distinguish them from asset pool parameters or variables. Portfolio constituents have their own systematic risk factor

$$F_t^{(\text{SP})} = \alpha F_{t-1}^{(\text{SP})} + \sigma W_t^{(\text{SP})} \quad (20)$$

which is, however, correlated with $F_t^{(\text{AP})}$ via the disturbances:

$$\text{Corr}\left(W_t^{(\text{SP})}, W_t^{(\text{AP})}\right) = \omega_2$$

Finally, LGD_{it} is given by

$$\text{LGD}_{it} = \frac{1}{1 + \exp\left(\mu^{(\text{SP})} + \tilde{\sigma}\sqrt{\omega_1}F_t^{(\text{SP})} + \tilde{\sigma}\sqrt{1 - \omega_1}E_{it}\right)} \quad (21)$$

All names have unit exposure⁴.

⁴For ΔVaR calculation we fixed the ratio of notionals of superportfolio and CDO

4 Stress Test Study

In this section we perform stress tests on our mezzanine tranches and study the evolution of their risk measures. We compare them with those of a bond under the same stress scenarios. To guarantee a proper comparison we match tranche and bond rating based on the cumulative 5-year tranche PD. That is, the 5-year tranche hitting probability implies a rating the term structure of which shall be used for the bond investment. In our case, the tranches have a hitting probability of 10.51%. This lies between BBB and BB on an S&P scale. We constructed a comparable bond term structure by interpolating cumulative PDs between BBB and BB.

As a final remark, the comparable bond's R_{it} needs to be driven by the same PD factor $F_t^{(\text{AP})}$ as the constituents of the asset pool.

4.1 Stress Scenarios

The stress settings we consider are to reveal adverse risk exposures as a result of extreme levels of critical model parameters. The following Table 3 shows the cases which we shall study. For the sake of clarity we stress only parameters relating to the asset pool and the comparable bond while the superportfolio model remains unchanged.

Case 0 is the base case which serves as a “normal” benchmark. It describes a neutral scenario with $F_0 \sim \mathcal{N}(0, 1)$, low asset correlation $\rho = 0.12$, and moderate autocorrelation $\alpha = 0.8$. The LGD parameters $\tilde{\sigma}$ and ω_1 follow the empirical estimates of Duellmann & Trapp (2004). The correlation of both PD factor processes, ω_2 , equals 0.6. F_0 , ρ , α , as well as the LGD parameters are identical in the asset pool and the superportfolio.

In a first set of cases we consider univariate stress scenarios. In case 1 we stress the systematic risk factors by fixing $F_0^{(\text{AP})}$ at the 0.1 quantile of the standard Gaussian.

In scenario 2 the correlation parameter $\rho^{(\text{AP})}$ is raised from 0.12 to 0.4⁵. Scenario 3 lifts the correlation between both PD factor processes from 0.6 to unity. Finally, in scenario 4, a sudden drop in recovery rates is assumed, so that mean LGD in the asset pool increases from 0.5 to 0.8.

In a second set of scenarios we consider multivariate (simultaneous) stress cases with increasing severity. Scenario 5 implies again a downturn cycle with initial value at the 0.1 quantile level of the factor distribution. In order to generate higher persistence of this adverse “environment” autocorrelation is increased from 0.8 to 0.95 which implies both higher conditional PDs as well as higher LGDs. Scenario 6 is equal to scenario 5 but in addition

tranche at $\frac{0.95}{0.05}$.

⁵As for the plausibility of this increase note that our base case asset correlation is at the lower bound of empirical estimates found in the literature.

has ω_2 , the correlation of the disturbances of the factor processes⁶ at its upper bound unity. This scenario pertains to strong systematic dependence between asset pool and bond portfolio. In scenario 7 we additionally lift asset correlation in both superportfolio and asset pool from 0.12 to 0.40. Finally, in scenario 8, scenario 6 is extended for a drop in mean recovery from 0.5 to 0.2.

Case	$F_0^{(AP)}$	$\rho^{(AP)}$	$\alpha^{(AP)}$	$\mu^{(AP)}$	$\bar{\sigma}^{(AP)}$	$\omega_1^{(AP)}$	ω_2
0	0	0.12	0.80	0	0.13	0.35	0.6
1	-1.28	0.12	0.80	0	0.13	0.35	0.6
2	0	0.40	0.80	0	0.13	0.35	0.6
3	0	0.12	0.80	0	0.13	0.35	1
4	0	0.12	0.80	-1.36	0.13	0.35	0.6
5	-1.28	0.12	0.95	0	0.13	0.35	0.6
6	-1.28	0.12	0.95	0	0.13	0.35	1
7	-1.28	0.40	0.95	0	0.13	0.35	1
8	-1.28	0.12	0.95	-1.36	0.13	0.35	1

Table 3: Parameter configurations.

4.2 Results

Subsequently, we show the effects of these scenarios on our tranches' risk measures. We start our discussion with tranche hazard rates.

t	0	1	2	3	4	5	6	7	8
Mezzanine Tranches									
1	0.0002	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0005	0.0014
2	0.0048	0.0101	0.0059	0.0001	0.0095	0.0030	0.0031	0.0685	0.1113
3	0.0192	0.0540	0.0251	0.0052	0.0600	0.0572	0.0566	0.2378	0.3701
4	0.0339	0.1019	0.0427	0.0223	0.1100	0.1743	0.1783	0.2845	0.4852
5	0.0481	0.1166	0.0490	0.0408	0.1383	0.2357	0.2454	0.2540	0.4748
Bond									
1	0.0097	0.0201	0.0033	0.0074	0.0079	0.0216	0.0218	0.0268	0.0219
2	0.0212	0.0364	0.0130	0.0177	0.0185	0.0437	0.0450	0.0589	0.0430
3	0.0269	0.0409	0.0206	0.0237	0.0255	0.0534	0.0525	0.0758	0.0533
4	0.0259	0.0393	0.0241	0.0264	0.0255	0.0524	0.0537	0.0791	0.0522
5	0.0250	0.0330	0.0245	0.0240	0.0235	0.0474	0.0473	0.0745	0.0485

Table 4: Hazard rates.

t	0	1	2	3	4	5	6	7	8	
Mezz. Tranches	5	0.1050	0.2571	0.1178	0.0673	0.2861	0.4068	0.4168	0.6212	0.8488
Bond	5	0.1050	0.1587	0.0828	0.0955	0.0970	0.2006	0.2021	0.2786	0.2010

Table 5: 5-year cumulative hitting probabilities.

⁶Note that these processes are always correlated unconditionally as long as their initial values coincide.

Hazard Rates and Hitting Probabilities In Table 4 the reader finds the hazard rates of tranches and comparable bonds and Table 5 shows the corresponding 5-year hitting probabilities. Scenarios are arranged in columns.

The dynamic evolution of both security types is obviously different. A common pattern across all scenarios (stress and base case) is that tranche hazard rates are lower than comparable bond rates initially. In a later period, however, tranche hazard rates increase strongly and exceed those of the corresponding bond. The reason for this “crossing hazard rates” effect is subordination melt-off: initially subordination protects the tranche from losses. Later on, when incurred losses have wiped out this protection layer, the risk of the tranche soars. The period when hazard rates “take-off” depends on the tranche’s seniority. For example, the mezzanine tranche is below the bond in the base case in the first three periods. But in $t = 4$ it is suddenly above the bond. By contrast, the hazard rate evolution of the bond is much smoother. Although not shown here directly, it is obvious that not only hazard rates but also cumulative PDs of tranches and bonds do cross in a similar fashion⁷.

Altogether, the impact of initial value and asset correlation stress entail the highest hazard rate growth in our stress design. In multivariate stress scenarios the effects are even stronger. Consequently, the take-off period of hazard rates decreases as stress increases. As noted above, cumulative PDs generally exceed those of bonds in final periods of almost all stress scenarios. For instance, due to our rating-matching 5-year PDs of tranche and bond are identical. However, in period five of scenario 8 the tranche hitting probability is more than four times as high as with the bond. Thus, 5-year hitting probabilities of the two tranches increase disproportionately.

To summarize, there are two important observations. First, the crossing hazard rates phenomenon underlines a different risk dynamics of CDOs and thus the relevance of dynamic modeling. Second, moving from the base case to scenario 8 shows the accelerated risk increase of tranches under stress.

Expected LGD Expected LGDs of tranche and bond differ significantly, as can be seen in Table 6. Bonds have LGDs in the region of 0.5 in all scenarios except for those where μ is stressed. In the latter case LGD shifts to about 0.8. By contrast, being cumulative mean LGD of both tranches have much more variation as they are cumulative. Obviously, mean LGD depends on the tranche width and, of course, on the amount of subordination. The larger a tranche, the lower its average LGD. Furthermore, we can see from scenario 4 and 8 that the LGD of our tranches is hardly influenced by the

⁷In Table 5 we see that 5-year cumulative PDs of the tranche are usually above those of the bond. Furthermore, we know that in $t = 1$ hazard rates and (cumulative) PDs coincide and in Table 4 we see that tranche rates are below those of the bond initially.

t	0	1	2	3	4	5	6	7	8
Mezzanine Tranche (Width: 3%)									
1	0.0657		0.0765		0.2414			0.1188	0.1943
2	0.3997	0.3432	0.4679	0.1419	0.3286	0.2094	0.1914	0.4268	0.3446
3	0.5020	0.4420	0.5753	0.3215	0.4440	0.3208	0.3146	0.5864	0.5492
4	0.5339	0.5263	0.6440	0.4165	0.5526	0.4505	0.4470	0.7078	0.7113
5	0.5621	0.5890	0.6847	0.4788	0.6313	0.5649	0.5584	0.7747	0.8173
Mezzanine Tranche (Width: 1%)									
1	0.1972		0.2296		0.5215			0.3564	0.5102
2	0.7181	0.6597	0.7530	0.4258	0.6284	0.5186	0.4831	0.7235	0.6502
3	0.7875	0.7447	0.8188	0.6149	0.7191	0.6385	0.6461	0.8315	0.8061
4	0.8006	0.7995	0.8515	0.7287	0.7984	0.7575	0.7538	0.8925	0.8923
5	0.8115	0.8332	0.8725	0.7669	0.8510	0.8293	0.8258	0.9220	0.9396
Bond (5-year cumulative PD: 10.51%)									
1	0.5318	0.5427	0.5379	0.5112	0.8030	0.5400	0.5428	0.5449	0.8162
2	0.5248	0.5392	0.5296	0.5118	0.8022	0.5401	0.5410	0.5433	0.8179
3	0.5235	0.5370	0.5322	0.5132	0.8030	0.5407	0.5395	0.5425	0.8182
4	0.5230	0.5354	0.5347	0.5152	0.8046	0.5406	0.5398	0.5439	0.8184
5	0.5231	0.5348	0.5350	0.5163	0.8055	0.5398	0.5394	0.5435	0.8187

Table 6: Expected LGD. Empty entries mean no loss observations.

average LGD in the asset pool.

Incremental VaR (IVaR) IVaR depends on the tranche’s thickness as well as its share of the superportfolio notional. A thinner tranche has usually higher risk contributions than a wider tranche. This is because a thin tranche has a higher probability of full loss than a thick tranche. A tranche possessing 5% of the superportfolio has maximum IVaR of 5% while a tranche with 10% superportfolio share has maximum IVaR of 10%.

Now, consider Table 7 showing IVaRs of cumulative loss. Obviously, the results are similar to those with hazard rates above. The risk contributions of the tranche are significantly lower in early periods but grow very quickly and ultimately exceed those of our benchmark bonds in later periods. In the base case IVaR of the 3% tranche is more than two times and of the 1% tranche it is more than three times as high as of the bond. In scenario 5 tranche IVaR₅ ratio of tranche and bond is almost four to one. Altogether, we record the highest IVaR levels in scenarios where LGD is shifted or the initial factor is lowered. This is the same result as with hazard rates. In multivariate stress scenarios the tranches contribute almost completely (i.e., with their full notional) to the VaR of the superportfolio. By contrast, the bond’s contribution is only 0.0270 in the worst scenario.

What does it mean when a tranche with 5% superportfolio notional has an IVaR of 5% in $t = 4$ or $t = 5$ (compare scenario 6-8)? It simply states that VaR increases by 5% when the tranche is added to the superportfolio. This amounts to almost surely default of the tranche. To summarize, the risk contribution of a CDO is lower than that of comparable bonds in early periods, even for bad scenarios. Later on, however, the risk may increase very quickly. In accordance with tranche hazard rates, this effect is even more pronounced in very bad stress scenarios. The order of stress scenarios

with increasing severity shows another fact. Both tranches contribute their full notional already in scenario 6. By contrast, the bonds' contributions are still far lower. However, in scenario 7 and 8 bond IVaR increases further while tranche IVaR has already reached its maximum. This underlines the increased stress sensitivity of tranches in comparison with bonds.

t	0	1	2	3	4	5	6	7	8
Mezzanine Tranche (Width: 3%)									
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002
2	0.0017	0.0031	0.0027	0.0000	0.0029	0.0004	0.0006	0.0373	0.0272
3	0.0079	0.0201	0.0150	0.0015	0.0194	0.0089	0.0167	0.0498	0.0463
4	0.0133	0.0318	0.0268	0.0182	0.0333	0.0287	0.0432	0.0500	0.0497
5	0.0208	0.0378	0.0345	0.0400	0.0393	0.0379	0.0493	0.0500	0.0500
Mezzanine Tranche (Width: 1%)									
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0004
2	0.0036	0.0076	0.0044	0.0001	0.0064	0.0016	0.0012	0.0473	0.0397
3	0.0119	0.0308	0.0235	0.0034	0.0291	0.0201	0.0344	0.0500	0.0488
4	0.0188	0.0387	0.0329	0.0383	0.0395	0.0385	0.0486	0.0500	0.0500
5	0.0259	0.0423	0.0387	0.0472	0.0431	0.0455	0.0497	0.0500	0.0500
Bond (5-year cumulative PD: 10.51%)									
1	0.0046	0.0101	0.0017	0.0037	0.0077	0.0096	0.0103	0.0129	0.0220
2	0.0048	0.0090	0.0044	0.0057	0.0110	0.0083	0.0095	0.0140	0.0194
3	0.0052	0.0103	0.0058	0.0088	0.0126	0.0086	0.0114	0.0179	0.0196
4	0.0069	0.0108	0.0087	0.0107	0.0128	0.0110	0.0134	0.0221	0.0224
5	0.0086	0.0108	0.0119	0.0117	0.0142	0.0108	0.0167	0.0247	0.0242

Table 7: Incremental VaR of cumulative loss.

5 Conclusion

In this article, we showed the particular relevance of stress tests for portfolios including CDOs. CDOs are more sensitive to systematic risk and thus also more “stress sensitive”, i.e., their risk contribution rises at higher pace than that of conventional bonds. Specifically, two key risk characteristics of CDOs have been found. First, we showed that the growth of tranche risk measures in later periods is significantly higher than with bonds in all scenarios. This underlines the relevance of a dynamic risk analysis. Isolated consideration of initially low tranche hazard rates is seriously misleading owing to accelerated risk increase in later periods. The latter applies in particular with buy-and-hold investments. Second, we found that the risk measures of the CDO increase also more quickly under stress than those of the bond. This emphasizes a clearly higher stress-sensitivity.

We explained why the tranche’s subordination is of major relevance for the timing of the transition. We showed how the tranche term structure directly translates into the evolution of VaR. In our simulations we found the greatest VaR impact with the initial level of the factor processes as well as with shifts in asset pool LGD. While the former is a very intuitive result, the latter is not so immediate but nevertheless very important. Indeed, it is of special interest currently as it establishes a connection between busting

bubbles and senior tranche loss.

Appendix

A Default Threshold Calculation

The covariance matrix of (R_{i1}, \dots, R_{it}) is necessary to back out the default thresholds c_{it} . In the following we derive Formula 10.

$$\begin{aligned}
Cov(R_{it}, R_{it'}) &= \mathbb{E}[R_{it} \cdot R_{it'}] - \mathbb{E}[R_{it}]\mathbb{E}[R_{it'}] \\
&= \rho\sigma^2 \sum_{j=0}^{t'-1} \alpha_1^{j+j+(t-t')} \mathbb{E}[W_{t-j}^2] + \rho\alpha_1^{t+t'} \mathbb{E}[F_0^2] + (1-\rho) \mathbf{1}_{\{t=t'\}} \\
&= \rho\sigma^2 \sum_{j=0}^{t'-1} \alpha_1^{2j+(t-t')} + \rho\alpha_1^{t+t'} + (1-\rho) \mathbf{1}_{\{t=t'\}} \\
&= \rho(1-\alpha_1^2) \sum_{j=0}^{t'-1} \alpha_1^{2j+(t-t')} + \rho\alpha_1^{t+t'} + (1-\rho) \mathbf{1}_{\{t=t'\}} \\
&= \rho \left(\sum_{j=0}^{t'} \alpha_1^{2j+(t-t')} - \sum_{j=1}^{t'} \alpha_1^{2j+(t-t')} \right) + (1-\rho) \mathbf{1}_{\{t=t'\}} \\
&= \rho\alpha_1^{|t-t'|} + (1-\rho) \mathbf{1}_{\{t=t'\}}
\end{aligned} \tag{22}$$

Note that this requires the unconditional factor process, i.e., $F_0 \sim \mathcal{N}(0, 1)$.

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